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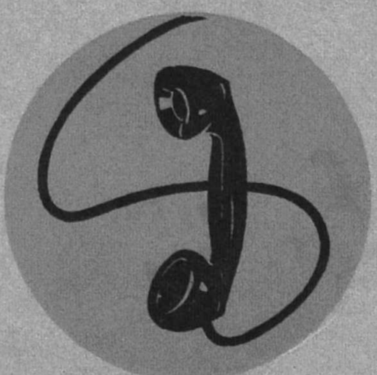
The reliability of telephone traffic switch counts



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The Reliability of Telephone Traffic Load Measurements by Switch Counts

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The switch count method of telephone traffic measurement is subject to sampling errors. The nature of these errors is discussed and formulas are derived which describe the extent of the errors under normally encountered traffic conditions.

INTRODUCTION

Of prime importance to the telephone traffic engineer is the determination of the busy season busy hour load carried by groups of trunks or other circuits of a telephone switching system. Three direct methods of measuring such loads are found in the field today. These are:

a. Peg Count and Holding Time Method

The number of calls carried by the circuit group during the observation period is counted. This number multiplied by the average holding time per call (in hundreds of seconds) and divided by the length of the observation period (in hours) gives an estimate of the group load in units of hundred-call-seconds per hour (CCS). The major drawback to this peg count method is that it requires a separate determination of the average holding time per call for the group under observation. R. I. Wilkinson[1] has analyzed the sources of errors of holding time measurements. In addition, correlation between load and holding time introduces an error which has not been studied.

b. Switch Count Method

At fixed intervals the circuit group is scanned and the number of busy circuits is counted. The total number of busy conditions counted divided by the number of scans is, then, an estimate of the load on the group in units of average simultaneous calls or erlangs*. This estimate is generally converted to CCS (1 erlang = 36 CCS) by traffic engineers since the

* The name "erlang" for average simultaneous call was adopted at a plenary meeting of the CCIF at Montreux in October, 1946.

load entries of most traffic tables are in terms of CCS. For theoretical studies the erlang is a more convenient unit and will be retained here.

c. Continuous Method

The busy condition of each circuit is represented by a fixed increment of electrical current through an ampere-hour meter. The instantaneous current is then analogous to the calls simultaneously present so that the meter, which integrates the current, may be calibrated to indicate hundred-call-seconds or erlang-hours directly. Although this method is potentially the most accurate, practical difficulties have limited its use.

In addition to these direct methods, there are several methods of indirect load measurement which, relying more heavily on traffic theory, make use of partial load indications, such as duration of group busy or the number of calls finding the group busy. Such measurements are less reliable than the direct measurements particularly when applied to underloaded groups.

This paper is concerned with the reliability of switch count load measurements since this method appears to have prospects of considerably wider adoption in the future. Main emphasis will be placed, both qualitatively and by the application of error formulas, on the relative effects of various measurement and traffic parameters on the accuracy of switch count measurements. Where long derivations of formulas are required they are deferred to the Appendix.

SOURCES OF ERROR

As has been described, switch count measurements yield the average number of calls found present when a group of circuits is scanned at fixed intervals during an observation period. Usually only that period of the day during which the load is greatest is of interest to the traffic engineer. Because the load during such periods also fluctuates from day to day, measurements of the loads for several days must be averaged to provide a useful load estimate.

There are two main sources of error, therefore, in switch count estimates of telephone traffic loads:

1. Each individual count of busy circuits is separated from the next by a time interval during which changes in load are not detected. Consequently, the load indicated by measurement may differ appreciably from the actual load carried. This difference can be decreased by decreasing the interval between scans.

2. Even if the load carried during a measurement period were known very accurately, it is still only a sample of the many loads that might be offered by the same source of traffic under statistically identical conditions. Therefore, the average of several load readings may be expected to be somewhat in error as an estimate of the true average of the traffic source. The latter will be referred to as the *source* load to distinguish it from the *carried* load.

Mechanical and human errors are likely to be present as well but, since they are not inherent in the switch count method, they will be neglected here.

SWITCH COUNT ERROR

As shown in the Appendix, for periods of observation which are relatively long with respect to average holding time made on traffic with certain assumed characteristics, the average error of switch counts in estimating traffic load *carried* in the same period is zero. The coefficient of variation of the error, which is the standard deviation of the error expressed in per cent of the traffic load carried, is given by:

$$V_x \doteq 100\sqrt{\left[r\operatorname{ctnh}\left(\frac{r}{2}\right) - 2\right]\frac{\bar{t}}{a'NT}} \qquad (1^*)$$

$$\operatorname{ctnh}\left(\frac{r}{2}\right) = \frac{1+e^{-r}}{1-e^{-r}} = \text{hyperbolic cotangent of } \frac{r}{2}$$

$$rc = T/\bar{t} > 20$$

where r = ratio of scan interval to holding time

$\bar{t}$ = average holding time

a' = *carried* load in erlangs

c = number of switch counts

T = length of observation period

N = number of observation periods

and where the following assumptions are made:

a. Calls originate individually and collectively at random.†

b. Holding times are exponentially distributed.

c. Congestion loss from the group is negligible.

* I have recently learned that these *carried* load formulas have been published by Conny Palm in *Tekniska Medelanden från Kungl. Telegrafstyrelsen,* 1941. nr. 7–9.

† See T. C. Fry, *Probability and Its Engineering Uses,* D. van Nostrand Co. Inc., New York, p. 216, for a definition of this condition.

As shown in the Appendix, this formula simplifies, when $r \leq 2$, to

$$V_x \doteq \frac{100}{c/T} \sqrt{\frac{1}{6a'NT\bar{t}}} \qquad (2)$$

where c/T = rate of scan in cycles per time unit. From equation (2) it is apparent that if the scan interval is of the order of a holding time, the error of an estimate of traffic *carried* is inversely proportional to the rate of scan and inversely proportional to the square root of average load, holding time and hours of observation. For example, take the case where switch counts are made during the busy hour, five minutes apart on a trunk group carrying calls with an average holding time of 3 minutes and an average load of 5 erlangs (180 CCS). What is the error in the estimated load *carried* if the readings for ten days are averaged? (Assume conditions (a), (b) and (c) are met.) We have

N = 10 observation periods

T = 1 hour

$\bar{t}$ = 1/20 hour

a' = 5 erlangs

c = 12 scans per observation period

$rc = T/\bar{t}$ = 20 average holding times per observation period

From equation (2) since $T/\bar{t} = 20$ and $r = T/c\bar{t} = 1.7$

$$V_x = \frac{100}{12} \sqrt{\frac{1}{6 \cdot 5 \cdot 10 \cdot 1 \cdot 1/20}} = 2.15\%$$

If, as proposed in the Appendix, it is assumed that the error has a normal distribution, there is 90 per cent assurance that observed values will fall within $1.64V_x$, or in the example within 3.52 per cent, of the true average*. Note that this error limit would be halved if the rate of scan were doubled or if four times as many hours of observation were taken.

The coefficient of variation of the switch count error for constant values of $T/\bar{t}$ as a function of r is plotted on Fig. 1 for one observation period of a one erlang load. For loads other than one erlang the coefficient of variation is found by dividing by $\sqrt{a'N}$. Thus in the example we have, using the dotted curve,

$$V_x = \frac{15}{\sqrt{5 \cdot 10}} = 2.1\%$$

* This assumes that a sufficient number of observations are taken so that a priori information may be neglected in making an estimate of the universe.

or more accurately using the solid curve,

$$V_x = \frac{16}{\sqrt{5 \cdot 10}} = 2.3\%$$

The error of using equation (2) is seen to be negligible for most purposes even when T/t is less than 20. The probability of an observation occuring within a given number of standard deviations is widely published for the normal curve. A few values are given below:

z	P_z Probability of exceeding $\pm z\sigma$ or $\pm zV$
0.6745	0.50
1.44	0.85
1.64	0.90
2.00	0.9545
3.00	0.9973

Fig. 2 is a plot for 40 observations of measured load vs carried load. Each observation was made for a half hour period on a panel line finder group with switch counts made at the start and middle of the period.

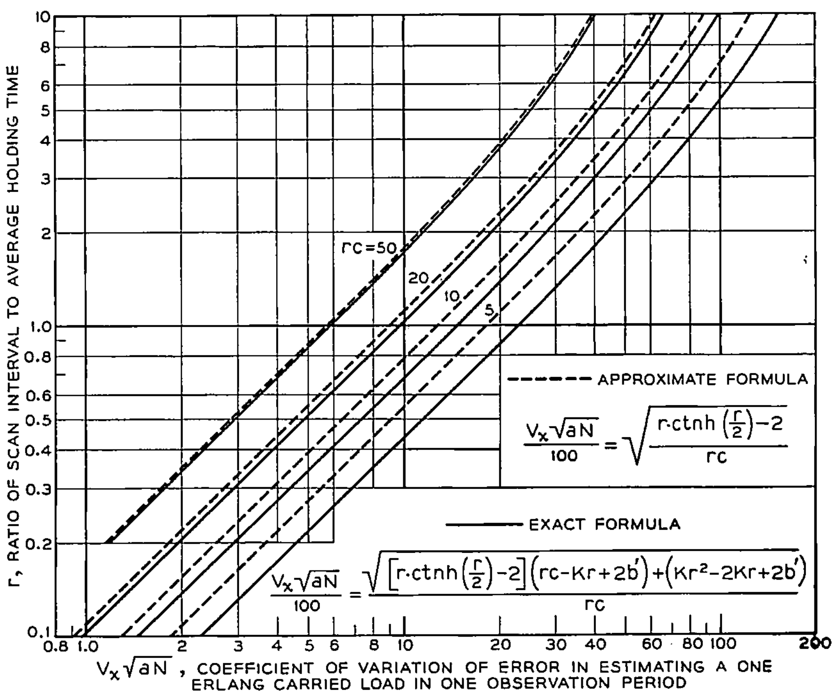

Fig. 1—Accuracy of switch count estimate of load actually carried.

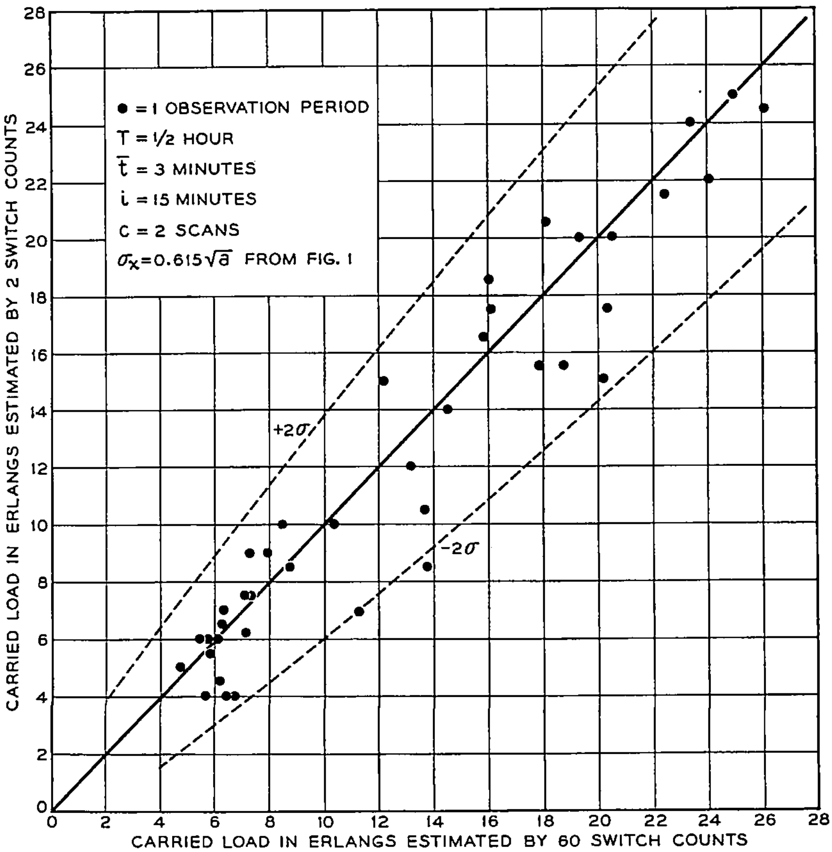

Fig. 2—Accuracy of switch count estimate of true average load.

This is compared with the average of switch counts made every 30 seconds which has a relatively negligible error. The average holding time per call for the group was 176 seconds. The accuracy of only two counts is surprisingly good and the observations are seen to lie satisfactorily between the 2σ limits.

ERROR OF TRAFFIC IN A GIVEN PERIOD AS AN ESTIMATE OF THE SOURCE LOAD

The average traffic carried in two different periods but generated by the same traffic source is subject to statistical variation. As a result, any measurement of load, even if measurement errors are eliminated, is only a sample of the wide range of traffic loads that might have been generated by the same source of traffic under identical circumstances.

J. Riordan has shown[2] that the standard deviation of the average traffic load for any one period is given by

$$\sigma_y = \sqrt{\frac{2a\bar{t}^2}{T^2}\left(\frac{T}{\bar{t}} - 1 + e^{-T/\bar{t}}\right)} \tag{3}$$

where a = the average source load

$\bar{t}$ = average holding time per call

T = length of observation period

(Assumptions are as before with an additional one that all periods are in statistical equilibrium)

When $\bar{t}/T \ll 1$ this reduces to the form also given by F. W. Rabe[3]

$$\sigma_y = \sqrt{\frac{2a\bar{t}}{T}} \tag{4}$$

or expressed in a per cent of the average

$$V_y = 100\sqrt{\frac{2\bar{t}}{aT}} \tag{5}$$

When N periods of length T are observed the coefficient of variation is reduced further to:

$$V_y = 100\sqrt{\frac{2\bar{t}}{aNT}} \tag{6}$$

In the example of the previous section,

$$N = 10$$

$$T = 1$$

$$\bar{t} = 1/20$$

$$a = 5$$

$$V_y = \sqrt{\frac{2 \cdot 1/20}{5 \cdot 10 \cdot 1}} = 4.47\%$$

COMBINATION OF ERRORS

Evidently if switch count readings are used to estimate the average which may be expected in other periods, the two errors described above should both be taken into account. The errors are probably correlated but this correlation is weak and at present no method of allowing for it is evident. Such a refinement would probably change the equation for standard deviation only slightly from that derived for the independent case; therefore independence will be assumed. The standard deviation of the sum of two independent variables is the square root of the sum of the squares of the component standard deviations:

$$\sigma_s = \sqrt{\sigma_x^2 + \sigma_y^2} \tag{7}$$

$$= \sqrt{\left[r\operatorname{ctnh}\left(\frac{r}{2}\right) - 2 \right] \frac{\bar{t}a'}{NT} + \frac{2\bar{t}a}{NT}} \tag{8}$$

Assuming $\frac{a'}{a}$ is approximately unity, that is, that carried load is approximately equal to source load,

$$V_s = 100 \sqrt{\frac{\bar{t}}{anT} r\operatorname{ctnh}\left(\frac{r}{2}\right)} \tag{9}$$

In the example given,

$$V_s = 4.96\%$$

There is, then, 90 per cent assurance that the source average is within $1.64 \times 4.96 = 8.1$ per cent of the observed average. Note that doubling the switch count rate (which halves the switch count error) reduces the total error only to 7.6 per cent (about 6.7 per cent improvement), while doubling the number of hours of observations reduces the error to 5.9 per cent (about 30 per cent improvement). Plots of the coefficient of variation of a one hour observation of a one erlang load versus scan rate for various average holding times are given in Fig. 3 for a wide range of holding times. The coefficient of variation of error in estimating other loads may be found from Fig. 3 by dividing the unit load coefficient by $\sqrt{aNT}$. In the example, the unit load coefficient is found, by entering Fig. 3 with $\bar{t} = 3$ minutes and rate of scan $= c/T = 12/1$ scan cycles per hour, to be 35.0 per cent. Dividing by $\sqrt{5 \cdot 1 \cdot 10}$ gives a coefficient of variation of 4.96 per cent as before. It is evident from Fig. 3 that increasing scan rates is not a universal way to improve the accuracy of *source* load estimates.

CHOICE OF SCAN RATES

What then governs the choice of scan rate? Evidently increasing the rate increases the accuracy of *carried* load estimates to any point desired. This is far from true if *source* load is being estimated. If the cost of making a scan is constant, increasing the number of observation periods and decreasing the scan rate will improve accuracy of *source* load estimates without changing measurement costs. The number of hours available for measuring, of course, limits this procedure, while the increase in accuracy becomes negligible as r becomes large. On the other hand, if the cost of each observation is only slightly affected by the cost of making additional scans, a high scan rate might be justified.

In applying the above relationships to traffic measurements, the usual question raised by the traffic engineer will be either how many hours of data need he take to be reasonably sure of his estimate or, conversely, how sure is he of an estimate based on available data. Assuming as before that the error distribution is normal, the per cent plus or minus error limits within which a proportion, P_z, of the estimates will fall is given by zV_s; the value of z corresponding to any selected P_z may be found from tables of the normal probability distribution. "Reasonably sure" is often taken to mean that there is 90 per cent assurance that the error does not exceed 5 per cent. When P_z is 0.90, z is 1.64, so that under this condition $1.64V_s = 0.05$, or $V_s = 0.0305$. Given scan rate and holding time, V_s is proportional to $1/\sqrt{aNT}$ according to equation (9) or Figure 3. When V_s is held constant, aNT is constant so that the plot of log NT against log a is linear, as shown in Figs. 4 and 5. The number of hours needed to meet any chosen reliability

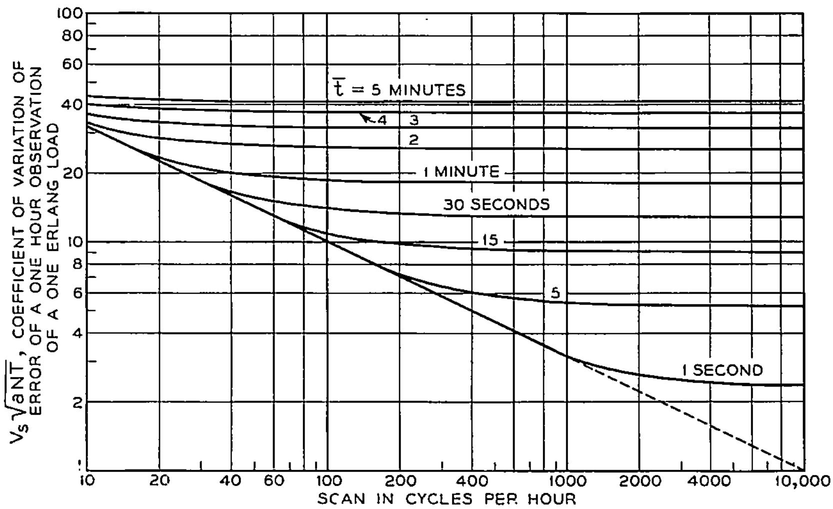

Fig. 3—Efficiency of switch counts for usage measurement.

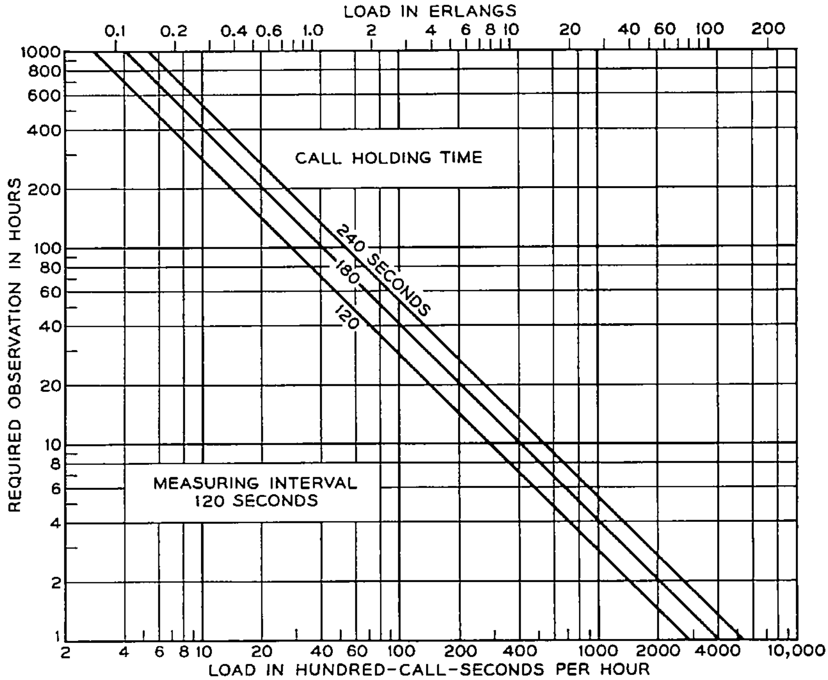

Fig. 4—Hours of measurement required for 90 per cent assurance that error in estimating *source* load does not exceed plus or minus 5 per cent when measuring interval is 120 seconds.

requirements may then be read directly from such graphs. In the second type of question, z, NT, scan rate and holding time are fixed so that zV_s is proportional to $1/\sqrt{a}$. Plotting log zV_s against log $\sqrt{a}$ again gives a linear plot as shown on Fig. 6.

In the numerical example above, the limits of error corresponding to 90 per cent assurance may be read from Fig. 6 which is plotted for the appropriate assurance, average holding time and scan interval. Reading the error limits at the point where the 10 hours measured line crosses 180 CCS (5 erlangs) gives $\pm$ 8.1 per cent as before. Fig. 5 may be entered to find the total number of hours required to reduce this error to 5 per cent. Reading at the point where the 180 second holding time line crosses 180 CCS gives 26 hours.

QUALITATIVE EXTENSION OF THEORETICAL APPROACH

The original traffic assumptions made in deriving the theoretical results above are:

a. Calls originate collectively and individually at random.

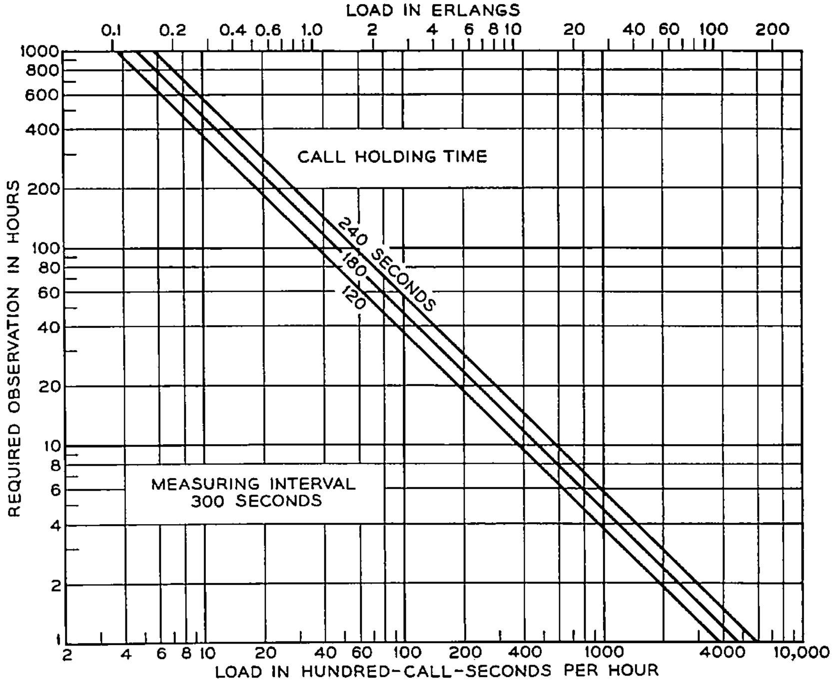

Fig. 5—Hours of measurement required for 90 per cent assurance that error in estimating *source* load does not exceed plus or minus 5 per cent when measuring interval is 300 seconds.

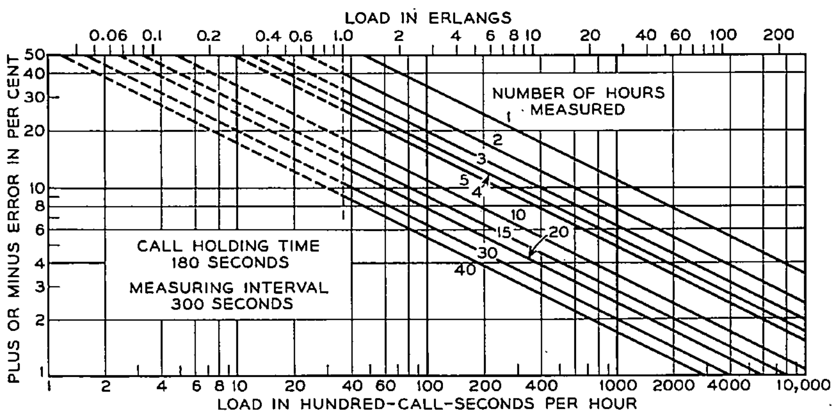

Fig. 6—Limits of error reached with 90 per cent assurance in estimating *source* load.

b. Holding times are exponentially distributed.
c. Congestion loss from the group is negligible.
d. Observation periods are in statistical equilibrium.

How do departures from these assumptions affect the reliability of usage measurements?

a. Holding Time Distribution

Experience in application of delay and loss formulas has shown that theories based on exponential holding times are often applicable to other holding time distribution cases which have a wide range. However, for a constant holding time distribution special theories often are called for. The average and standard deviation of switch count estimates of *carried* load when holding time is constant, are given in part 2 of the Appendix. It is shown there that for estimates of *carried* load,

$$\bar{x} = 0$$

$$r \geq 1 \qquad V_x = 100 \sqrt{\frac{\bar{t}}{a'NT}(r - 1)} \tag{10}$$

$$r \leq 1 \quad \text{minimum } V_x = 0 \quad (r = 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \text{etc.})$$

$$\text{maximum } V_x = 100 \frac{r}{2} \sqrt{\frac{\bar{t}}{a'NT}} \quad (r = \tfrac{2}{3}, \tfrac{2}{5}, \tfrac{2}{7}, \text{etc.}) \tag{11}$$

Since constant holding times found in practice are often very short, the case of $r \geq 1$ is the most likely to be met. For all values of r greater than one, the error given by formula (1) for exponential holding times is somewhat greater than the error given by formula (10) for constant holding times, so use of formula (1) for the constant holding time case is conservative. For values of r less than 1, the error is an oscillating function of r. The coefficient of variation varies from zero to 23 per cent above that for exponential holding times. Where r may not be accurately known the formula for exponential holding times again seems appropriate.

In making estimates of the *source* load when the holding time is constant, if $r \geq 1$, each scan is uncorrelated with any other, since no call can be counted twice, and may be considered a random sample of traffic. There are a total of Nc scans which have an average scan of a and standard deviation $\sqrt{a}$. The average error in estimating a is, therefore:

$$\bar{s} = 0$$

with coefficient of variation

$$V_s = 100 \sqrt{\frac{1}{aNc}} = 100 \sqrt{\frac{\bar{t}}{aNT}r} \tag{12}$$

Equation (12) may also be derived with the procedure used for equation (9) using $\sigma_s^2 = \sigma_x^2 + \sigma_y^2$. For values of r large enough to make $\operatorname{ctnh}\left(\frac{r}{2}\right) \doteq 1$ equation (12) is approached by equation (9). For smaller values of r (but with r still greater than 1), V_s for constant holding times is less than V_s for exponential holding times. When $r = 1$, there is no *carried* load error. For values of r less than 1, the coefficient of variation of error in estimating *source* load average will vary from

$$\sqrt{\frac{\bar{t}}{aNT}} \quad \text{to} \quad \sqrt{\frac{\bar{t}}{aNT}\left(1 + \frac{r^2}{2}\right)}$$

depending on the exact value of r. It is interesting to note that V_s for $r = 0.5$ is the same as for $r = 1.125$.

b. Loss

The effect of loss in the group depends upon the disposition of the lost calls. In general, accuracy in measuring *carried* load increases with increased loss because under these circumstances fewer load changes occur between scans. This is evident in the extreme case of a group which is 100 per cent loaded; a single switch count gives a correct reading for any length period. Obviously load readings at 100 per cent occupancy are not very useful in estimating *offered* loads since the amount of lost load cannot even be guessed at. However, in the cases of lost calls held (Poisson) or cleared (Erlang B), the *offered* load may be estimated from the *carried* load (less and less accurately as occupancy increases) and in the case of lost calls delayed the *offered* and *carried* loads are likely to be the same even at high occupancies. With high loss, therefore, estimates of *source* load are subject to errors not considered in deriving equation (99); however, switch count error in estimating *carried* load will be materially less than predicted by equation (1).

c. Random Call Origination

On trunk groups which are alternate routes, calls may no longer be considered as originating at random. The resultant grouping of call originations will tend to decrease the accuracy of switch count measurements in estimating *carried* load; however, there is a corresponding decrease in accuracy in estimating the *source* load from the *carried* load so that accuracy in estimating *carried* load may be less worthwhile.

d. Statistical Equilibrium

Statistical equilibrium may be thought of as the absence of trends in subscriber calling rates or holding times with the passage of time. The effect of trends on switch count accuracy in measuring *carried load* is very small except where the changes in traffic level are frequent and abrupt with respect to the scan frequency. Such traffic behavior is rare.

Trends within the busy hour complicate the problem of estimating the average *source load*. However, it can be shown that if the trends are small (in the order of 10 per cent to 20 per cent) little error is introduced by assuming that no trend exists. Large trends (in the order of 100 per cent), however, may indicate that the traffic source is so unstable that more hours of traffic data should be taken in order to insure that the sample is representative.

Trends from day to day do not affect the *source* load estimates in the same way as within hour trends. The *source* loads are seldom exactly the same on any two days although in most offices a load pattern is repeated from week to week. The traffic engineer may be interested in the average *source* load of either a typical week day in the busy season or, sometimes, of the average of the two highest days in the week. As long as the *source* load of each particular day remains close to the average for that day of the week, the general average for several different days of the week, will be known with about the same accuracy as if they had all come from a common source. If, however, there is no stable pattern in the *source* load, a third error in estimating the average is generated. There is some difficulty in determining whether or not variations in load, as indicated by measurements, are due to sampling variations or to an unstable source. Quality control methods might be used to detect instability but gathering and processing sufficient data for such an analysis might prove uneconomical. In general, if a traffic engineer feels that his *source* load is unstable he will need more hours of data than indicated by formula (9) to meet a given criterion of reliability.

CONCLUSIONS

A theoretical approach to the problem of the accuracy of switch count measurements in estimating *carried* load and average *source* load has been explored. It is believed that the assumptions made are satisfied sufficiently often in practice to enable fairly wide application of the results of this exploration to traffic measurements. However, it should be kept in mind that where the assumptions are clearly not valid, special allowances will need to be made. In any case, the confidence placed in usage measurements by a traffic engineer is a function of his experience and judgment. It is hoped that the results of this study will add to the knowledge essential to sound traffic engineering.

APPENDIX

DERIVATION OF SWITCH COUNT ERROR IN ESTIMATING CARRIED LOAD—WITH EXPONENTIAL HOLDING TIMES

This derivation is based on a similar derivation by R. I. Wilkinson[1]. However, since load rather than holding time is of interest here, the emphasis has been somewhat shifted.

Assume that switch count measurements are being taken on traffic with:

a. Calls originated individually and collectively at random
b. Exponentially distributed holding times
c. Negligible loss

Let i = interval between scans

$\bar{t}$ = average holding time

a' = traffic carried, in erlangs

T = length of observation period

$r = \dfrac{i}{\bar{t}}$ = number of holding times in a scan interval

$c = \dfrac{T}{i}$ = number scans in observation period

$rc = \dfrac{T}{\bar{t}}$ = number of holding times in observation period

N = number of observation periods.

Consider that the observation period begins with the first scan and ends i time units after the last scan. It is desired to find the error in estimating the true load carried by averaging the number of circuits found busy on each scan. Following Wilkinson's method we will first estimate the error of the switch count method in measuring the contribution of a single call to total usage and then modify it to take account of n calls. Calls of two types must be considered, those originating outside the interval and extending into it, Type I, and those originating within the interval, Type II. Both types may be subdivided depending on whether or not they extend beyond the end of the observation period. These are indicated in Fig. 7. Only that part of a call which falls within the observation period contributes to the usage of that period. First the error made by switch counts in measuring this contribution will be derived.

Type I

Consider a call which is already in progress at the start of the observation period. Its duration beyond that point, according to theory, will be exponentially distributed about an average of $\bar{t}$.

If this duration, t, is between 0 and i, the call will be counted once (a measured contribution of i erlang hours) and a positive error of $x = i - t$ will be made. The same error will be made if $t = 2i - x$ so that the call

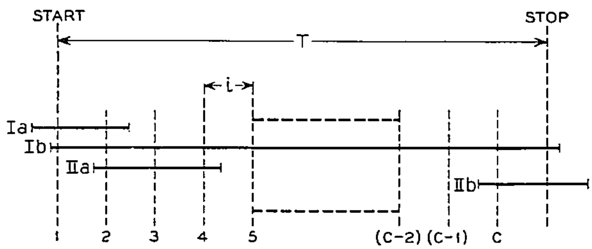

Fig. 7—Graphical indication of the two types of calls with their two subdivisions.

is counted twice and so forth. Summing all the ways of making an error x, we have:

$$P(x)\,dx = f(i - x) + f(2i - x) + \cdots f(ci - x) \qquad (1)$$

where $f(i - x)$ is the probability of $t = i - x$ and

$$f(t) = \frac{1}{\bar{t}}\,e^{-t/\bar{t}}\,dx$$

Calls lasting beyond ci neither start nor end in the observation period so that their contribution is measured without error. For these:

$$P(0) = P(t \geq ci) = e^{-rc} \qquad (2)$$

Therefore:

$$P_{x>0}(x)\,dx = \frac{1}{\bar{t}}\,e^{-\frac{i - x}{\bar{t}}}\,dx$$

$$+ \frac{1}{\bar{t}}\,e^{-\frac{2i - x}{\bar{t}}}\,dx + \cdots + \frac{1}{\bar{t}}\,e^{-\frac{ci - x}{\bar{t}}}\,dx \qquad (3)$$

Letting

$$y = \frac{x}{t}; \qquad e^{-cr} = b = 1 - b'; \qquad K = \sum_{n=0}^{c-1} e^{-nr} = \frac{b'}{1 - e^{-r}}$$

$$P_{x>0}(x)\,dx = e^{y}e^{-r}K\,dy \tag{4}$$

The moment generating function $M_I(\alpha)$ of y is:

$$\begin{aligned} M_I(x) &= \int_0^r P(y)\,dy\,e^{\alpha y} + e^{-rc} \\ &= b + K\frac{e^{r\alpha} - e^{-r}}{1 + \alpha} \end{aligned} \tag{5}$$

Neglecting terms of order higher than α^2,

$$M_I(\alpha) = 1 + \alpha(rK - b') + \frac{\alpha^2}{2}(Kr^2 + 2b' - 2rK) \tag{6}$$

Type II

Calls of Type II may have either positive or negative errors given by:

$$\begin{aligned} P_{x\leq 0}(x)\,dx = \frac{i + x}{i}[&f_0(-x) + f_1(i - x) + f_2(2i - x) \\ &+ \cdots + f_{c-1}((c - 1)i - x)] \\ &+ g_0(-x) + g_1(i - x) + g_2(2i - x) + \cdots + g_{c-1}[(c - 1)i - x] \end{aligned} \tag{7}$$

$$P_{x\geq 0}(x)\,dx = \frac{i - x}{i}[f_1(i - x) + f_2(2i - x) + \cdots + f_c(ci - x)]$$

where $f_n(ni - x)$ = probability that a Type II call has length $ni - x$ and ends before the end of the observation period.

$$= \frac{1}{t}e^{-\frac{ni-x}{t}} \cdot \frac{c - n}{c}\,dx$$

$g_n(ni - x)$ = probability that a Type II call starts $ni - x$ before the end of the observation period and ends after the end of the observation period.

$$= \frac{1}{T}e^{-\frac{ni-x}{t}}\,dx$$

Equation (7) becomes:

$$
\begin{aligned}
P_{x\leq 0}(x)\,dx &= \left[\frac{i+x}{i}\sum_{n=0}^{c-1}\frac{1}{\bar{t}}\,e^{-\frac{n i - x}{\bar{t}}}\cdot\frac{c-n}{c}+\sum_{n=0}^{c-1}\frac{1}{\bar{T}}\,e^{-\frac{n i - x}{\bar{t}}}\right]dx \\
P_{x\geqq 0}(x)\,dx &= \left[\frac{i-x}{i}\sum_{n=1}^{c}\frac{1}{\bar{t}}\,e^{-\frac{n i - x}{\bar{t}}}\cdot\frac{c-n}{c}\right]dx
\end{aligned}
\tag{8}
$$

Letting

$$\frac{x}{\bar{t}} = y, \qquad K = \sum_{n=0}^{c} e^{-nr} \text{ as before}$$

and noting that

$$\sum_{n=0}^{c-1} ne^{-nr} = \frac{K-cb}{1-e^{-r}}$$

$$
\begin{aligned}
P_{x\leq 0}(x)\,dx &= e^{y}\left[\left(1+\frac{y}{r}\right)\left(K-\frac{1}{c}\,\frac{e^{-r}K-cb}{1-e^{-r}}\right)+\frac{K}{rc}\right]dy \\
P_{x\geq 0}(x)\,dx &= e^{-r}e^{y}\left[\left(1-\frac{y}{r}\right)\left(K-\frac{1}{c}\,\frac{K-cb}{1-e^{-r}}\right)\right]dy
\end{aligned}
\tag{9}
$$

The moment generating function of this pair of equations is the sum of their separate m.g.f.'s:

$$
\begin{aligned}
M_{II}(\alpha) &= \int_{-r}^{0} P_{y\leq 0}(y)e^{\alpha y}\,dy + \int_{0}^{r} P_{y\geq 0}(y)\,dy\;e^{\alpha y}\,dy \\
&= \frac{rc+K+c-2\,\dfrac{c-Ke^{-r}}{1-e^{-r}}+\dfrac{c-K}{1-e^{-r}}\,e^{\alpha r}+\dfrac{(c-K)e^{-r}}{1-e^{-r}}\,e^{-\alpha r}+\alpha(K+rc-e^{-r}Ke^{-\alpha r})}{rc(1+\alpha)^2}
\end{aligned}
\tag{10}
$$

Neglecting terms of order higher than α^2,

$$
\begin{aligned}
&M_{II}(\alpha) \\
&= \frac{1}{rc}\left\{rc+\alpha(b'-rK)+\frac{\alpha^2}{2}\left[\left(r\operatorname{ctnh}\left(\frac{\alpha}{2}\right)-2\right)(rc-rK+2b')\right]\right\}
\end{aligned}
\tag{11}
$$

Now the number of Type I calls present in an observation is a variable—with average "a" and a Poisson distribution. Similarly the number of Type II calls is a variable, independent of the number of Type I calls, with an average of "$a\,\frac{T}{\bar{t}}$" or "arc" and a Poisson distribution. According to the laws governing the compounding of variables the moment generating function of the sum of n variables y, when n is also variable with generating function $G(t)$, is $G(M(\alpha))$ where $M(\alpha)$ is the moment generating function of y.

The generating function of a Poisson variable with average "a" is e^{-a+at} so that

$$\begin{aligned} G(M_I(\alpha)) &= e^{-a+aM_I(\alpha)} \\ G(M_{II}(\alpha)) &= e^{-arc+arcM_{II}(\alpha)} \end{aligned} \tag{12}$$

These independent variables may be added by multiplying their moment generating functions to give the m.g.f. of the total measurement error of the *carried* load

$$M(\alpha) = e^{-(a+arc)+aM_I(\alpha)+arcM_{II}(\alpha)} \tag{13}$$

From (6), (11) and (13) the following parameters are found:

$$\bar{y} = 0 \tag{14}$$

$$\sigma_y^2 = arc\left[\left(r\operatorname{ctnh}\left(\frac{r}{2}\right) - 2\right)\left(1 - \frac{K}{c} + \frac{2b'}{rc}\right) + \left(\frac{rK}{c} - 2\frac{K}{c} + \frac{2b'}{rc}\right)\right]$$

If, now, rc is sufficiently large

$$\sigma_y \doteq \sqrt{arc\left[r\operatorname{ctnh}\left(\frac{r}{2}\right) - 2\right]} \tag{15}$$

It is more convenient to deal with the standard deviation expressed as per cent of the *carried* erlang load, the coefficient of variation. This is done by multiplying both sides of equation (15) by $\bar{t}$ to convert the time dimension from holding times to hours, dividing by T to convert from erlang-hours to erlangs, dividing by a' to convert to proportion of *carried* load, and multiplying by 100 to convert to per cent. Assuming $\frac{a}{a'^2}$ is approximately $\frac{1}{a'}$:

$$V_x \doteq 100\sqrt{\frac{\left[r\operatorname{ctnh}\left(\frac{r}{2}\right) - 2\right]\bar{t}}{a'T}}$$

When N observations are made this reduces further to

$$V_x \doteq 100\sqrt{\frac{\bar{t}}{a'NT}\left[r\operatorname{ctnh}\left(\frac{r}{2}\right) - 2\right]}$$

$$\text{Now } \operatorname{ctnh}(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots \qquad (x^2 < \pi^2)$$

and for $r \leq 2$

$$\frac{r/2}{3} \gg \frac{(1/2)^3}{45}$$

Therefore

$$r\text{ctnh}\left(\frac{r}{2}\right) \doteq 2 + \frac{r^2}{6}$$

$$V_x \doteq 100\sqrt{\frac{\bar{t}}{a'NT}\frac{r^2}{6}} = \frac{100}{c/T}\sqrt{\frac{1}{6a'NT\bar{t}}} \qquad (r \leq 2) \quad (16)$$

The error in *carried* load may be considered as the sum of a large number of independent errors. Its distribution may, therefore, be expected to approach the normal distribution. Comparison of the third and fourth moments of the normal distribution with those of the error distribution (which may be obtained from equation (13)) show good agreement for values of a' greater than 1.

DERIVATION OF SWITCH COUNT ERROR IN ESTIMATING CARRIED LOAD WITH CONSTANT HOLDING TIMES

Wilkinson has shown[1] that, for constant holding time, switch count error in measuring the holding time of one call has an average

$$\bar{x} = 0$$

and standard deviation

$$\sigma_x = \sqrt{-x_1 x_2}$$

where $T \gg \bar{t}$

$T \gg i$

$x_1 =$ negative error

$x_2 =$ positive error

Divide the problem into two parts:

1. For $r > 1$

$$x_1 = -\bar{t}$$

$$x_2 = i - \bar{t}$$

$$\sigma_x = \sqrt{\bar{t}i - \bar{t}^2} = \bar{t}\sqrt{r - 1} \qquad (17)$$

2. For $r \leq 1$

$$\text{Min. } \sigma_x = 0 \qquad \text{for} \qquad r = 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \text{etc.}$$

$$\text{Max. } \sigma_x = \sqrt{\frac{i}{2} - \frac{i}{2}} = \frac{i}{2} = \bar{t}\,\frac{r}{2} \tag{18}$$

$$\text{for} \qquad r = \tfrac{2}{3}, \tfrac{2}{5}, \tfrac{2}{7}, \text{etc.}$$

Expressing this error in terms of carried load and proceeding as in Part I of the Appendix

$$1.\ r > 1 \qquad V_x = 100 \sqrt{\frac{\bar{t}}{a'NT}(r-1)} \tag{19}$$

$$2.\ r \leq 1 \text{ Min. } V_x = 0$$

$$\text{Max. } V_x = 100\,\frac{r}{2}\sqrt{\frac{\bar{t}}{aNT}} \tag{20}$$

$$= \frac{100}{c/T}\sqrt{\frac{1}{4a'NT\bar{t}}}$$

Equation (20) compares favorably with the exponential holding time coefficient of variation of error of

$$\frac{100}{c/T}\sqrt{\frac{1}{6a'NT\bar{t}}}$$

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